Noncommutative space-time models

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Abstract

The FRT quantum Euclidean spaces O_q^N are formulated in terms of Cartesian generators. The quantum analogs of N-dimensional Cayley-Klein spaces are obtained by contractions and analytical continuations. Noncommutative constant curvature spaces are introduced as a spheres in the quantum Cayley-Klein spaces. For $N = 5$ part of them are interpreted as the noncommutative analogs of (1+3) space-time models. As a result the quantum (anti) de Sitter, Newton, Galilei kinematics with the fundamental length and the fundamental time are suggested.

1 Introduction

Space-time is a fundamental conception which underlines the most significant physical theories. Therefore analysis of a possible space-time models (or kinematics) has the fundamental meaning for physics. Possible commutative kinematics were described in [1] on the level of Lie algebras. From the point of view of geometry these kinematics are realized as constant curvature spaces, which can be obtained from the spherical space by contractions and analytical continuations known as Cayley-Klein (CK) scheme [2].

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New possibility for construction of the noncommutative space-time models is provided by quantum groups and quantum vector spaces [3]. The quantum Poincaré group related to the κ-Poincaré algebra as well as the κ-Minkowski kinematics were suggested [4]–[6]. A general formalism that allows the construction of field theory in κ-Minkowski space-time was developed [7]. An approach connected with quantum deformation of Lie algebras is mainly used in these papers.

The purpose of our paper is to obtain the noncommutative (quantum) analogs of the possible kinematics starting with the quantum Euclidean space. CK scheme of contractions and analytical continuations was developed in Cartesian basis whereas the standard quantum group theory [3] was formulated in a different skew-symmetric one. Therefore first of all this theory is reformulated in the Cartesian basis, then the noncommutative analogs of constant curvature spaces (CCS) including fiber (or flag) spaces are investigated and some of them are interpreted as noncommutative kinematics.

2 Commutative kinematics

Classical four-dimensional space-time models can be obtained [2] by the physical interpretation of the orthogonal coordinates of the most symmetric spaces, namely constant curvature spaces. All 3^N N-dimensional CCS are realized on the spheres

$$S_N(j) = \{\xi_1^2 + j_1^2\xi_2^2 + \ldots + (1,N+1)^2\xi_{N+1}^2 = 1\}, \tag{1}$$

where

$$(i,k) = \prod_{l=\min(i,k)}^{\max(i,k)-1} j_l, \quad (k,k) \equiv 1, \tag{2}$$

and each of parameters j_k takes the values $1, \iota_k, i,\ k = 1,\ldots,N$. Here ι_k are nilpotent generators $\iota_k^2 = 0$, with commutative law of multiplication $\iota_k\iota_m = \iota_m\iota_k \neq 0,\ k \neq m$.

The intrinsic Beltrami coordinates $x_k = \xi_{k+1}\xi_1^{-1},\ k = 1,2,\ldots,N$ present the coordinate system in CCS, which coordinate lines $x_k = const$ are geodesic. CCS has positive curvature for $j_1 = 1$, negative for $j_1 = i$ and it is flat for $j_1 = \iota_1$. For a flat space the Beltrami coordinates coincide with the Cartesian ones. Nilpotent values $j_k = \iota_k,\ k > 1$ correspond to a fiber (flag) spaces and imaginary values $j_k = i$ correspond to pseudo-Riemannian spaces.

Classical $(1+3)$ kinematics [1] are obtained from CCS for $N=4,\ j_1 = 1, \iota_1, i,\ j_2 = \iota_2, i,\ j_3 = j_4 = 1$ if one interprets x_1 as the time axis $t = \xi_2 \xi_1^{-1}$ and the rest as the space axes $r_k = \xi_{k+2}\xi_1^{-1}, k = 1,2,3$. The standard de Sitter kinematics $S_4^{(-)}$ with constant negative curvature is realized for $j_1 = j_2 = i$, anti de Sitter kinematics $S_4^{(+)}$ with positive curvature — for $j_1 = 1,\ j_2 = i$. Relativistic flat Minkowski kinematics M_4 appears for $j_1 = \iota_1,\ j_2 = i$. Nonrelativistic Newton $N_4^{(\pm)}$ and Galilei G_4 kinematics correspond to $j_2 = \iota_2$, $j_1 = 1, i$ and $j_1 = \iota_1$, respectively.

3 Quantum Cayley-Klein spaces

Let us remind the definition of the quantum vector space [3]. An algebra $O_q^N(\mathbf{C})$ with generators $x_1, \ldots, x_N$ and commutation relations

$$\hat{R}_q(x \otimes x) = qx \otimes x - \frac{q - q^{-1}}{1 + q^{N-2}} x^t C x W_q, \qquad (3)$$

where $\hat{R}_q = PR_q,\ Pu \otimes v = v \otimes u,\ \forall u, v \in \mathbf{C}^n,\ W_q = \sum_{i=1}^N q^{\rho_{i'}} e_i \otimes e_{i'}$,

$$x^t C x = \sum_{i,j=1}^{N} x_i C_{ij} x_j = \epsilon x_{n+1}^2 + \sum_{k=1}^{n} \left(q^{-\rho_k} x_k x_{k'} + q^{\rho_k} x_{k'} x_k \right), \qquad (4)$$

$\epsilon = 1$ for $N = 2n+1$, $\epsilon = 0$ for $N = 2n$ and vector $(e_i)_k = \delta_{ik},\ \ i, k = 1, \ldots, N$ is called the algebra of functions on N-dimensional quantum Euclidean space (or simply the quantum Euclidean space) $O_q^N(\mathbf{C})$.

The matrix C has non-zero elements only on the secondary diagonal. They are equal to unit in the commutative limit $q = 1$. Therefore the quantum vector space $O_q^N(\mathbf{C})$ is described in a skew-symmetric basis, where for $q = 1$ the invariant form $inv = x^t C_0 x$ is given by the matrix C_0 with the only unit non-zero elements on the secondary diagonal.

In the case of kinematics, the most natural basis is the Cartesian basis, where the invariant form $inv = y^t y$ is given by the unit matrix I. The transformation from the skew-symmetric basis x to the Cartesian basis y is described by $y = D^{-1}x$, where matrix D is a solution of the equation $D^t C_0 D = I$. This equation has many solutions. Take one of these, namely

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & -i\tilde{C}_0 \\ 0 & \sqrt{2} & 0 \\ \tilde{C}_0 & 0 & iI \end{pmatrix},\ N = 2n+1, \qquad (5)$$

where $\tilde{C}_0$ is the $n \times n$ matrix with real units on the secondary diagonal and all other elements equal to zero. For $N = 2n$ the matrix D is given by (5) without the middle column and row. The matrix (5) provides one of the possible combinations of the quantum group structure and the CK scheme of group contractions. All other similar combinations are given by the matrices $D_\sigma = DV_\sigma$, obtained from (5) by the right multiplication on the matrix $V_\sigma \in M_N$ with elements $(V_\sigma)_{ik} = \delta_{\sigma_i,k}$, where $\sigma \in S(N)$ is a permutation of the N-th order.

We derive the quantum Cayley-Klein spaces with the same transformation of the Cartesian generators $y = \psi\xi, \ \psi = \mathrm{diag}(1, (1,2), \dots, (1,N)) \in M_N$, as in commutative case [2]. The transformation $z = Jv$ of the deformation parameter $q = e^z$ should be added in quantum case. The commutation relations of the Cartesian generators of the quantum N-dimensional Cayley-Klein space are given by the equations

$$\hat{R}_\sigma(j)\xi \otimes \xi = e^{Jv}\xi \otimes \xi - \frac{2shJv}{1 + e^{Jv(N-2)}}\xi^t C_\sigma(j)\xi W_\sigma(j),$$

where

$$\hat{R}_\sigma(j) = \Psi^{-1}(D_\sigma \otimes D_\sigma)^{-1}\hat{R}_q(D_\sigma \otimes D_\sigma)\Psi, \quad W_\sigma(j) = \Psi^{-1}(D_\sigma \otimes D_\sigma)^{-1}W_q,$$

$$C_\sigma(j) = \psi D_\sigma^t C D_\sigma \psi = \psi V_\sigma^t D^t C D V_\sigma \psi, \ \ \Psi = \psi \otimes \psi. \tag{6}$$

The explicit form of commutation relations see [9]. The multiplier J is chosen as $J = \bigcup_{k=1}^{n}(\sigma_k, \sigma_{k'})$. This is the minimal multiplier, which guarantees the existence of the Hopf algebra structure for the associated quantum group $SO_v(N; j; \sigma)$. The "union" $(\sigma_k, \sigma_p) \bigcup (\sigma_m, \sigma_r)$ is understood as the first power multiplication of all parameters j_k, which occur at least in one multiplier (σ_k, σ_p) or (σ_m, σ_r), for example, $(j_1 j_2) \bigcup (j_2 j_3) = j_1 j_2 j_3$.

Quantum orthogonal Cayley-Klein sphere $S_v^{(N-1)}(j;\sigma)$ is obtained as the quotient of $O_v^N(j;\sigma)$ by $inv(j) = \xi^t C_\sigma(j)\xi = 1$. The quantum analogs of the intrinsic Beltrami coordinates on this sphere are given by the sets of independent right or left generators

$$r_{\sigma_i - 1} = \xi_{\sigma_i}\xi_1^{-1}, \quad \hat{r}_{\sigma_i - 1} = \xi_1^{-1}\xi_{\sigma_i}, \quad i = 1, \dots, N, \quad i \neq k, \quad \sigma_k = 1. \tag{7}$$

In the case of quantum Euclidean spaces $O_q^N(\mathbf{C})$ the use of different D_σ for $\sigma \in S(N)$ makes no sense, because all similarly obtained quantum spaces

are isomorphic. However the situation is radically different for the quantum Cayley-Klein spaces. In this case the Cartesian generators ξ_k are multiplied by $(1,k)$ and for nilpotent values of all or some parameters j_k this isomorphism of quantum vector spaces is destroyed. The necessity of using different D_σ arises as well if there is some physical interpretation of generators. In this case physically different generators may be confused by permutations σ, for example, time and space generators of kinematics. Mathematically isomorphic kinematics may be physically nonequivalent.

4 Quantum kinematics

For $N=5$ the thorough analysis of the multiplier $J=(\sigma_1,\sigma_5)\cup(\sigma_2,\sigma_4)$, which appears in the transformation of the deformation parameter $z=Jv$, and commutation relations (6) of the quantum space generators for different permutations allowed to find two permutations giving a different J and a physically nonequivalent kinematics, namely $\sigma_0=(1,2,3,4,5)$ and $\sigma'=(1,4,3,5,2)$.

In order to clarify the relation with the standard Inonu–Wigner contraction procedure [8], the mathematical parameter j_1 is replaced by the physical one $\tilde{j}_1 T^{-1}$, and the parameter j_2 is replaced by ic^{-1}, where $\tilde{j}_1=1,i$. The limit $T\to\infty$ corresponds to the contraction $j_1=\iota_1$, and the limit $c\to\infty$ corresponds to $j_2=\iota_2$. The parameter T is interpreted as the curvature radius and has the physical dimension of time $[T]=[\text{time}]$, the parameter c is the light velocity $[c]=[\text{length}][\text{time}]^{-1}$.

As far as the generator ξ_1 does not commute with others, it is convenient to introduce right and left time $t=\xi_2\xi_1^{-1},\ \hat{t}=\xi_1^{-1}\xi_2$ and space $r_k=\xi_{k+2}\xi_1^{-1},\ \hat{r}_k=\xi_1^{-1}\xi_{k+2},\ k=1,2,3$ generators. The reason for this definition is the simplification of expressions for commutation relations of quantum kinematics. The commutation relations of the independent generators are obtained (see [9] for details) in the form

$$S_v^{4(\pm)}(\sigma_0)=\{t,\mathbf{r}|\ \hat{t}r_1=\hat{r}_1 t\cos\frac{\tilde{j}_1 v}{cT}+i\hat{r}_1 r_2\frac{1}{c}\sin\frac{\tilde{j}_1 v}{cT},$$

$$\hat{t}r_2-\hat{r}_2 t=-2i\hat{r}_1 r_1\frac{1}{c}\sin\frac{\tilde{j}_1 v}{2cT},\ \ \hat{t}r_3=\hat{r}_3 t\cos\frac{\tilde{j}_1 v}{cT}-it\frac{cT}{\tilde{j}_1}\sin\frac{\tilde{j}_1 v}{cT},$$

$$\hat{r}_1 r_2=\hat{r}_2 r_1\cos\frac{\tilde{j}_1 v}{cT}-i\hat{t}r_1 c\sin\frac{\tilde{j}_1 v}{cT},\ \ \hat{r}_p r_3=\hat{r}_3 r_p\cos\frac{\tilde{j}_1 v}{cT}-ir_p\frac{cT}{\tilde{j}_1}\sin\frac{\tilde{j}_1 v}{cT}\},\quad (8)$$

$$S_v^{4(\pm)}(\sigma') = \{t, \mathbf{r}| \ \hat{r}_k t = \hat{t} r_k \cosh\frac{\tilde{j}_1 v}{T} - i r_k \frac{T}{\tilde{j}_1}\sinh\frac{\tilde{j}_1 v}{T},$$

$$\hat{r}_2 r_1 = \hat{r}_1 r_2 \cosh\frac{\tilde{j}_1 v}{T} - i\hat{r}_1 r_3 \sinh\frac{\tilde{j}_1 v}{T}, \ \ \hat{r}_1 r_3 = \hat{r}_3 r_1 \cosh\frac{\tilde{j}_1 v}{T} - i\hat{r}_2 r_1 \sinh\frac{\tilde{j}_1 v}{T},$$

$$\hat{r}_2 r_3 - \hat{r}_3 r_2 = 2i\hat{r}_1 r_1 \sinh\frac{\tilde{j}_1 v}{2T}\}. \tag{9}$$

In the case of the identical permutation σ_0, deformation parameter v for the system units, where $\hbar = 1$, has the physical dimension of length $[v] = [cT] = [\text{length}]$ and may be interpreted as the fundamental length. For the permutation σ', the quantum (anti) de Sitter kinematics (9) are characterized by the fundamental time $[v] = [\text{time}]$. Recall that the same physical dimensions of the deformation parameter have been obtained for the quantum algebras $so_v(3; j; \sigma)$ and corresponding $(1+1)$ kinematics for a different permutations [10].

In the zero curvature limit $T \to \infty$ two quantum Minkowski kinematics are obtained

$$M_v^4(\sigma_0) = \{t, \mathbf{r}| \ \ [t, r_p] = 0, \ [r_3, t] = ivt, \ [r_2, r_1] = 0, \ [r_3, r_p] = ivr_p, \ p = 1, 2, \},$$

$$M_v^4(\sigma') = \{t, \mathbf{r}| \ \ [t, r_k] = ivr_k, \ \ [r_i, r_k] = 0, \ i, k = 1, 2, 3\}. \tag{10}$$

The first one is isomorphic to the tachyonic κ-Minkowski kinematics, the second one to the standard κ-deformation [4]–[6]. For both κ-Minkowski kinematics in the system units $\hbar = c = 1$ the deformation parameter $\Lambda = \kappa^{-1}$ has the physical dimension of length and is interpreted as the fundamental length. But in the system units $\hbar = 1$ the deformation parameter has different dimensions, namely v is the fundamental length for $M_v^4(\sigma_0)$ and v is the fundamental time for $M_v^4(\sigma')$.

As far as the commutation relations (10) do not depend on c, they do not change in the limit $c \to \infty$, therefore the generators of the quantum Galilei kinematics $G_v^4(\sigma_0)$ and $G_v^4(\sigma')$ are subject of the same commutation relations.

In the nonrelativistic limit $c \to \infty$ there are two noncommutative analogs of the Newton kinematics $(p = 1, 2)$

$$N_v^{4(\pm)}(\sigma_0) = \{t, \mathbf{r}| \ [t, r_p] = 0, \ \ [r_3, t] = ivt(1 + \tilde{j}_1^2 \frac{t^2}{T^2}),$$

$$[r_1, r_2] = 0, \ \ [r_3, r_p] = ivr_p(1 + \tilde{j}_1^2 \frac{t^2}{T^2})\},$$

$$N_v^{4(\pm)}(\sigma') = \{t, \mathbf{r}| \ [t, r_k] = i(r_k + \frac{\tilde{j}_1^2}{T^2} t r_k t) \frac{T}{\tilde{j}_1} \tanh \frac{\tilde{j}_1 v}{T},$$

$$r_2 r_1 = r_1 r_2 \cosh \frac{\tilde{j}_1 v}{T} - i r_1 r_3 \sinh \frac{\tilde{j}_1 v}{T},$$

$$r_1 r_3 = r_3 r_1 \cosh \frac{\tilde{j}_1 v}{T} - i r_2 r_1 \sinh \frac{\tilde{j}_1 v}{T}, \ \ [r_2, r_3] = 2 i r_1^2 \sinh \frac{\tilde{j}_1 v}{2T}\}, \tag{11}$$

where in the last case the deformation parameter is not transformed under contraction. The multiplier T^{-1} appears as the result of the physical interpretation of the quantum space generators. For nonzero curvature kinematics commutation relations of generators depend on c and are different for relativistic and nonrelativistic cases, unlike Minkowski and Galilei kinematics.

5 Conclusion

We have reformulated the quantum Euclidean space O_q^N in Cartesian coordinates and then used the standard trick with real, complex, and dual numbers in order to define the quantum Cayley-Klein spaces $O_q^N(j;\sigma)$ uniformly. Noncommutative constant curvature spaces are generated by the Beltrami coordinates on spheres $S_v^{N-1}(j;\sigma)$. The different combinations of quantum structure and CK scheme are described with the help of permutations σ. As a result for $N = 5$, the quantum deformations of (anti) de Sitter, Minkowski, Newton and Galilei kinematics are obtained. We have found two types of the noncommutative space-time models with fundamental length and fundamental time.

The quantum Galilei kinematics have the same commutation relations as the quantum Minkowski kinematics. In other words, the quantum deformations of the flat kinematics are identical, whereas for nonzero curvature kinematics commutation relations of generators are different for relativistic and nonrelativistic cases.

Noncommutative kinematics are obtained by the interpretation of some mathematical constructions associated with quantum groups and quantum spaces. Noncommutativity of space and time generators appear at the distance comparable with the fundamental length or at the time interval comparable with the fundamental time. The deformation parameter is free parameter of these models. Which type of the model is more appropriate and

what is the value of deformation parameter, i.e. the values of fundamental length and fundamental time, are questions of experimental study.

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